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Basic Maths and Vectors

SOLUTIONS

BEGINNER'S BOX-1

1. (i) $\tan(-30°) = -\tan 30° = -\frac{1}{\sqrt{3}}$

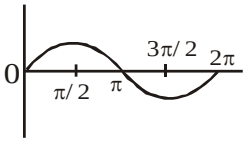
(ii) $\sin(120°) = \sin(180° - 60°) = \sin 60° = \frac{\sqrt{3}}{2}$

(iii) $\sin(135°) = \sin(90° + 45°) = \cos 45° = \frac{1}{\sqrt{2}}$

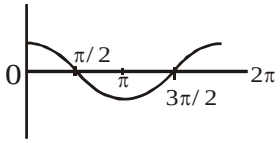
(iv) $\cos(150°) = \cos(180° - 30°) = -\cos(30°)$

$= -\frac{\sqrt{3}}{2}$

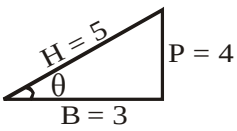
(v) $\sin(270°) = (-1)$ (from graph)



(vi) $\cos 270° = 0$



2. $\sec\theta = \frac{5}{3}$



we know that, $\sec\theta = \frac{H}{B} = \frac{5}{3}$

$\cos\theta = \frac{3}{5}$;

we know 3, 4, 5 are pythogorous triplets

$\therefore$ P = 4

i.e. H : B : P = 5 : 3 : 4

$\therefore$ $\sin\theta = \frac{4}{5}$ $\quad$ $\operatorname{cosec}\theta = \frac{5}{4}$

$\cos\theta = \frac{3}{5}$ $\quad$ $\sec\theta = \frac{5}{3}$

$\tan\theta = \frac{4}{3}$ $\quad$ $\cot\theta = \frac{3}{4}$

BEGINNER'S BOX-2

1. Let P(8, –4) and Q(0, a) be two points, distance PQ is 10.

$\therefore$ According to distance formula

$PQ = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$= \sqrt{(0-8)^2+(a+4)^2}$

$\Rightarrow PQ = \sqrt{64+(a^2+16+8a)}$

$\Rightarrow 10 = \sqrt{80+a^2+8a}$

$\Rightarrow a^2 + 8a + 80 = 100 \Rightarrow a^2 + 8a = 20$

$\Rightarrow a^2 + 10a - 2a - 20 = 0$

$\Rightarrow (a - 2)(a + 10) = 0 \Rightarrow (a = 2)$ and $(a = -10)$

$\therefore a = 2$

2. $P(x_1, y_1, z_1) = (0, -1, +1)$

and $Q(x_2, y_2, z_2) = (3, 3, 13)$

using distance formula

$PQ = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

$= \sqrt{9+16+144} = \sqrt{169} = 13$

3. Slope $m = \frac{\Delta y}{\Delta x} = \frac{y_2-y_1}{x_2-x_1} = \tan\theta$

$m = \frac{\Delta y}{\Delta x} = \frac{(0-2)}{(3-0)} = \frac{-2}{3}$

BEGINNER'S BOX-3

1. (i) $y = x^{7/2}$

$\frac{dy}{dx} = \frac{d}{dx}(x^{7/2}) = \frac{7}{2}x^{\frac{7}{2}-1} = \frac{7}{2}x^{5/2}$

(ii) $y = x^{-3}$ $\quad\therefore\quad$ $\frac{dy}{dx} = -3x^{-4}$

(iii) $y = x$ $\quad\therefore\quad$ $\frac{dy}{dx} = 1$

(iv) $y = x^5 + x^3 + 4x^{1/2} + 7$

$\frac{dy}{dx} = 5x^4 + 3x^2 + 2x^{-1/2} + 0$

$= 5x^4 + 3x^2 + 2x^{-1/2}$

(v) $y = 5x^4 + 6x^{3/2} + 9x$

$\frac{dy}{dx} = 20x^3 + 9x^{3/2-1} + 9$

$= 20x^3 + 9x^{1/2} + 9$

(vi) $y = ax^2 + bx + c \Rightarrow \frac{dy}{dx} = 2ax + b$

(vii) $y = 3x^5 - 3x - \frac{1}{x}$

$\frac{dy}{dx} = 15x^4 - 3 + \frac{1}{x^2}$

2. $s = t^2 + 5t + 3 \Rightarrow \frac{ds}{dt} = 2t + 5$

3. $s = ut + \frac{1}{2}at^2 \Rightarrow \frac{ds}{dt} = u + at$

4. $A = 3t^2 + 7\ cm^2$

$\frac{dA}{dt} = \frac{d}{dt}(3t^2 + 7) = 6t$

at $t = 5, \frac{dA}{dt} = 30 cm^2/s$

5. $A = \pi r^2$

$\frac{dA}{dr} = \frac{d}{dr}(\pi r^2) = 2\pi r$

6. (i) $\frac{d}{dx}(x-1)(2x+5)$

$= (x-1)\frac{d}{dx}(2x+5) + (2x+5)\frac{d}{dx}(x-1)$

$= (x-1)(2) + (2x+5)(1) = 2x - 2 + 2x + 5$

$= 4x+3$

(ii) $\frac{d}{dx}\frac{1}{(2x+1)} = \frac{0-2}{(2x+1)^2} = \frac{-2}{(2x+1)^2}$

(iii) $\frac{d}{dx}\left(\frac{3x+4}{4x+5}\right) = \frac{(4x+5)(3)-(3x+4)(4)}{(4x+5)^2}$

$= \frac{-1}{(4x+5)^2}$

(iv) $\frac{d}{dx}\left(\frac{x^2}{x^3+1}\right) = \frac{(x^3+1)(2x)-x^2(3x^2)}{(x^3+1)^2}$

$= \frac{2x^4+2x-3x^4}{(x^3+1)^2} = \frac{2x-x^4}{(x^3+1)^2}$

BEGINNER'S BOX-4

1. (i) $\int x^{15}dx = \frac{x^{15+1}}{15+1} + C = \frac{x^{16}}{16} + C$

(ii) $\int x^{-3/2}dx = \frac{x^{-\frac{3}{2}+1}}{-\frac{3}{2}+1} + C = -2x^{-\frac{1}{2}} + C$

(iii) $\int (3x^{-7} + x^{-1})dx = \frac{3x^{-6}}{-6} + \log_e x + C$

$= \frac{-x^{-6}}{2} + \log_e x + C$

(iv) $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx$

$= \int \left(x + \frac{1}{x} + 2\right)dx = \frac{x^2}{2} + \log_e x + 2x + C$

(v) $\int \left(x + \frac{1}{x}\right)dx = \frac{x^2}{2} + \log_e x + C$

(vi) $\int \left(\frac{a}{x^2} + \frac{b}{x}\right)dx = \int (ax^{-2} + bx^{-1})dx$

$= \frac{ax^{-2+1}}{-2+1} + b\log_e x + C = -\frac{a}{x} + b\log_e x + C$

BEGINNER'S BOX-5

1. (i) $\int_R^\infty \frac{GMm}{x^2}dx = GMm\int_R^\infty x^{-2}dx$

$= GMm\left[\frac{x^{-1}}{-1}\right]_R^\infty = \frac{GMm}{R}$

(ii) $\int_{r_1}^{r_2} \frac{-kq_1q_2}{x^2}dx$

$= -kq_1q_2\left[\frac{x^{-1}}{-1}\right]_{r_1}^{r_2} = kq_1q_2\left[\frac{1}{r_2} - \frac{1}{r_1}\right]$

(iii) $\int_u^v Mv\,dv = M\left[\frac{v^2}{2}\right]_u^v = \frac{1}{2}M(v^2 - u^2)$

(iv) $\int_0^\infty x^{-1/2}dx = \left[\frac{x^{1/2}}{\frac{1}{2}}\right]_0^\infty = \infty$

(v) $\int_0^{\pi/2} \sin x\,dx = [-\cos x]_0^{\pi/2} = 1$

(vi) $\int_0^{\pi/2} \cos x\,dx = [\sin x]_0^{\pi/2} = 1$

(vii) $\int_{-\pi/2}^{\pi/2} \cos x\,dx = [\sin x]_{-\pi/2}^{\pi/2} = 2$

BEGINNER'S BOX-6

1. (i) $10x^2 - 27x + 5 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{27 \pm \sqrt{(-27)^2 - 4(10)(5)}}{20}$$

$$= \frac{27 \pm 23}{20} \Rightarrow x = \frac{5}{2}, \frac{1}{5}$$

(ii) $(pq)x^2 - (p^2 + q^2)x + (pq) = 0$

$$x = \frac{(p^2+q^2) \pm \sqrt{(p^2+q^2)^2 - 4(pq)(pq)}}{2pq}$$

$$= \frac{(p^2+q^2) \pm (p^2-q^2)}{2pq} \Rightarrow x = \frac{p}{q}, \frac{q}{p}$$

BEGINNER'S BOX-7

1. $1 + 2 + 3 + \ldots\ldots\ldots\ldots + 50$

$$s_n = \frac{n}{2}[2a + (n-1)d] = \frac{50}{2}[2 + (49)1]$$

$= 25\,[51] = 1275$

2. $1^2 + 2^2 + 3^2 \ldots\ldots\ldots\ldots + 10^2$

$$s = \frac{n(n+1)(2n+1)}{6} = \frac{(10)(11)(21)}{6} = 385$$

BEGINNER'S BOX-8

1. $$s_\infty = \frac{a}{1-r} = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$$

2. $$F_{net} = GMm\left[\frac{1}{r^2} + \frac{1}{2r^2} + \frac{1}{4r^2} + \ldots\ldots\infty\right]$$

$$= GMm\left[\frac{1/r^2}{1-\frac{1}{2}}\right] = \frac{2GMm}{r^2}$$

BEGINNER'S BOX-9

1. Let the magnitudes of forces be A and B.

then according to question $A - B = 10$

$\Rightarrow A = 10 + B$ .....(1)

and $\sqrt{A^2 + B^2} = 50$

$\Rightarrow A^2 + B^2 = 2500$ .....(2)

Solving equation number (1) & (2)

$A = 40N$ and $B = 30N$

Therefore the magnitudes of forces are 40N & 30N

2. Let θ be the angle between $\hat{a}$ & $\hat{b}$ then

$$\left|\hat{a} - \hat{b}\right| = \sqrt{1^2 + 1^2 - 2(1)(1)\cos\theta} = \sqrt{2}$$

$$\Rightarrow \cos\theta = 0 \Rightarrow \theta = 90^\circ \text{ or } \frac{\pi}{2} \text{ radian}$$

$$\left|\hat{a} + \sqrt{3}\,\hat{b}\right| = \sqrt{1 + 3 + 2\sqrt{3}\cos\theta} = \sqrt{1+3} = 2$$

3. Unit vector along $\left(\vec{A} - \vec{B}\right) = \frac{\vec{A} - \vec{B}}{\left|\vec{A} - \vec{B}\right|}$

$$= \frac{\left(3\hat{i} + 2\hat{j}\right) - \left(2\hat{i} + 3\hat{j} - \hat{k}\right)}{\left|\left(3\hat{i} + 2\hat{j}\right) - \left(2\hat{i} + 3\hat{j} - \hat{k}\right)\right|} = \frac{\hat{i} - \hat{j} + \hat{k}}{\sqrt{3}}$$

4. $\because \left|\hat{A} + \hat{B}\right| = \sqrt{2} \Rightarrow 2 \times 1 \times \cos\frac{\theta}{2} = \sqrt{2}$

$$\Rightarrow \cos\frac{\theta}{2} = \frac{1}{\sqrt{2}} \Rightarrow \frac{\theta}{2} = 45^\circ \Rightarrow \theta = 90^\circ$$

$$\left|\hat{A} - \hat{B}\right| = 2 \times 1 \times \sin\frac{\theta}{2} = 2 \times \sin 45^\circ = \sqrt{2}$$

BEGINNER'S BOX-10

1. $\because$ $A_x = 4$, $A_y = 6$ so $A_x + B_x = 10$ and

$A_y + B_y = 9$

(a) $B_x = 10 - 4 = 6m$ and $B_y = 9 - 6 = 3m$

(b) length $= \sqrt{B_x^2 + B_y^2} = \sqrt{36+9} = \sqrt{45}m$

(c) $\theta = \left(\frac{B_y}{B_x}\right)\tan^{-1} = \tan^{-1}\left(\frac{3}{6}\right) = \tan^{-1}\left(\frac{1}{2}\right)$

2. Let α, β & γ are the angles of vector $\left(5\hat{i} + 2\hat{j} + 6\hat{k}\right)$ from x, y & z-axis respectively.

then $$\cos\alpha = \frac{A_x}{A} = \frac{5}{\left|5\hat{i} + 2\hat{j} + 6\hat{k}\right|} = \frac{5}{\sqrt{65}},$$

$$\cos\beta = \frac{A_y}{A} = \frac{2}{\left|5\hat{i} + 2\hat{j} + 6\hat{k}\right|} = \frac{2}{\sqrt{65}}$$

$$\cos\gamma = \frac{A_z}{A} = \frac{6}{\left|5\hat{i} + 2\hat{j} + 6\hat{k}\right|} = \frac{6}{\sqrt{65}}$$

The sum of squares of directional cosines of this vector

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{5^2 + 2^2 + 6^2}{65} = \frac{65}{65} = 1$$

3. Resultant $(\vec{R})$ of

$\vec{A}\,\&\,\vec{B} = \vec{A} + \vec{B} = (6\hat{i} - 6\hat{j} + 5\hat{k}) + (\hat{i} + 2\hat{j} - \hat{k})$

$= 7\hat{i} - 4\hat{j} + 4\hat{k}$

$\therefore$ unit vector parallel to resultant $(\hat{R})$

$= \dfrac{\vec{R}}{|\vec{R}|} = \dfrac{7\hat{i} - 4\hat{j} + 4\hat{k}}{|7\hat{i} - 4\hat{j} + 4\hat{k}|} = \dfrac{7\hat{i} - 4\hat{j} + 4\hat{k}}{9}$

BEGINNER'S BOX-11

1. $\because a = b = 1$ so $a^2 + b^2 + 2ab\cos\theta = (\sqrt{3})^2 = 3$

$\Rightarrow 1 + 1 + 2\cos\theta = 3 \Rightarrow \cos\theta = \dfrac{1}{2}$

$(\vec{a} - \vec{b}) \cdot (2\vec{a} + \vec{b}) = 2a^2 - b^2 + ab\cos\theta - 2ab\cos\theta$

$= 2 - 1 - \dfrac{1}{2} = \dfrac{1}{2}$

2. $\because$ $\vec{A}$ & $\vec{B}$ are perpendicular to each other

$\therefore \vec{A} \cdot \vec{B} = 0$

$\Rightarrow (4\hat{i} - 2\hat{j} + 4\hat{k}) \cdot (-4\hat{i} + 2\hat{j} + \alpha\hat{k}) = 0$

$\Rightarrow -16 - 4 + 4\alpha = 0 \Rightarrow \alpha = 5$

3. $\because (\hat{a} + 2\hat{b})$ is perpendicular to $(5\hat{a} - 4\hat{b})$

$\therefore (\hat{a} + 2\hat{b}) \cdot (5\hat{a} - 4\hat{b}) = 0$

$\Rightarrow 5\hat{a}\cdot\hat{a} - 4\hat{a}.\hat{b} + 10\hat{b}.\hat{a} - 8\hat{b}.\hat{b} = 0$

$\Rightarrow 5 + 6\hat{a}\cdot\hat{b} - 8 = 0 \Rightarrow 6\hat{a}\cdot\hat{b} = 3$

$\Rightarrow \hat{a}\cdot\hat{b} = \dfrac{1}{2} \Rightarrow \cos\theta = \dfrac{1}{2} \Rightarrow \theta = 60°$

4. (a) Let the angle between $\vec{A}$ and $\vec{B}$ is θ, then

$\cos\theta = \dfrac{\vec{A}\cdot\vec{B}}{AB}$

$= \dfrac{(2\hat{i} - 2\hat{j} - \hat{k}) \cdot (\hat{i} + \hat{j})}{|2\hat{i} + 2\hat{j} - \hat{k}||\hat{i} + \hat{j}|} = \dfrac{0}{3\sqrt{2}} = 0 \Rightarrow \theta = 90°$

(b) Resultant vector

$(\vec{R}) = \vec{A} + \vec{B} = (2\hat{i} - 2\hat{j} - \hat{k}) + (\hat{i} + \hat{j}) = 3\hat{i} - \hat{j} - \hat{k}$

$\therefore$ projection of resultant on x-axis = 3

5. Vector component of $\vec{a}$ in the direction of $\vec{b}$

$= (a\cos\theta)\,\hat{b} = \left(\dfrac{ab\cos\theta}{b}\right)\dfrac{\vec{b}}{b} = \dfrac{\vec{a}.\vec{b}}{|\vec{b}|^2}\vec{b}$

$= \dfrac{(2\hat{i} + 3\hat{j}).(\hat{i} + \hat{j})}{[\sqrt{1^2 + 1^2}]^2}(\hat{i} + \hat{j})$

$= \dfrac{2\times1 + 3\times1}{2}(\hat{i} + \hat{j}) = \dfrac{5}{2}(\hat{i} + \hat{j})$

BEGINNER'S BOX-12

1. (a) Area of the parallelogram $= |\vec{A} \times \vec{B}| =$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 0 \\ 0 & 1 & 2 \end{vmatrix}$$

$= |2\hat{i} - 6\hat{j} + 3\hat{k}| = \sqrt{2^2 + (-6)^2 + 3^2} = 7$ units

(b) Unit vector perpendicular to both $\vec{A}$ & $\vec{B}$

$\hat{n} = \dfrac{\vec{A}\times\vec{B}}{|\vec{A}\times\vec{B}|} = \dfrac{2\hat{i} - 6\hat{j} + 3\hat{k}}{7} = \dfrac{2}{7}\hat{i} - \dfrac{6}{7}\hat{j} + \dfrac{3}{7}\hat{k}$

2. $\sqrt{3}|\vec{A}\times\vec{B}| = \vec{A}.\vec{B}$

$\sqrt{3}\,AB\sin\theta = AB\cos\theta$

$\tan\theta = 1/\sqrt{3}$

$\theta = 30°$

3. If vectors $\vec{A}$ & $\vec{B}$ are sides of a triangle then its area $= \dfrac{1}{2}|\vec{A}\times\vec{B}| = \dfrac{AB}{4}$

$\therefore \dfrac{1}{2}AB\sin\theta = \dfrac{AB}{4} \Rightarrow \sin\theta = \dfrac{1}{2}$

$\theta = 30°$ or $150°$

4. As $|\vec{A} \times \vec{B}|^2 = |\vec{A}|^2 |\vec{B}|^2 - (\vec{A}.\vec{B})^2$

$\therefore 8^2 = 2^2 \times 5^2 - (\vec{A}.\vec{B})^2$

$\therefore (\vec{A}.\vec{B})^2 = 100 - 64 = 36$

$\therefore \vec{A}.\vec{B} = \pm 6$